Khaja Bandanawaz University

Faculty of Engineering and Technology

B.E. First Semester

Question Bank

Subject: Calculus and Linear Algebra (19KBMAT11)

Module 1

1. With usual notation, prove that $\tan \emptyset = r\frac{d\theta}{dr}$.
2. Find the angle between the radius vector and tangent for the curve $r = a(1 + \cos\theta)$ and also find the slope of the tangent at $\theta = \pi/3$.
3. Find the angle between the radius vector and tangent for the curve $r^m = a^m(\cos m\theta + \sin m\theta)$.
4. Find the angle between the curves $r^2 \sin 2\theta = 4\ and\ r^2 = 16\sin 2\theta$.
5. Find the angle of intersection between the curves $r = a(1 + \sin\theta)$ and $r = a(1 - \cos\theta)$.
6. Show that the curves $r = a(1 + \cos\theta)$ and $r = b(1 - \cos\theta)$ cut each other orthogonally.
7. Prove that the following pairs of curves $r^n = a^n \cos n\theta\ \ and\ r^n = b^n \sin n\theta$ intersect orthogonally.
8. With usual notation, prove that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}(\frac{dr}{d\theta})^2$.
9. Obtain the pedal equation of the curve $r^n = a^n \cos n\theta$
10. Obtain the pedal equation of the curve $r^n = a(1 + \cos n\theta)$.
11. Find the radius of curvature of the curve $r^n = a^n \cos n\theta$.
12. Find the radius of curvature of the curve $y^2 = \frac{4a^2(2a-x)}{x}$ where it cuts the x-axis.
13. Find the radius of curvature of the curve $x^3 + y^3 = 3axy$ at $(\frac{3a}{2}, \frac{3a}{2})$.
14. Expand $\sin x$ in ascending powers of $(x - \frac{\pi}{2})$ upto the 4th degree term.
15. Using Maclaurins expansion, prove that $\sqrt{1 + \sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots$

Module 2

1. If $u = (x^2 + y^2 + z^2)^{-1/2}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.
2. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$ and hence show that $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 u = -\frac{9}{(x+y+z)^2}$.
3. If $z = xy^2 + x^2y$ where $x = at^2, y = 2at$, find $\frac{dz}{dt}$. Also verify the result by direct substitution.
4. If $u = \tan^{-1}(\frac{y}{x})\ where\ x = e^t - e^{-t}\ and\ y = e^t + e^{-t}, find\ \frac{du}{dt}$.
5. If $u = f(y - z, z - x, x - y)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

6. If $z = f(x,y)$ where $x = r\cos\theta, y = r\sin\theta$, show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$.
7. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.
8. If $u = x^2 + y^2 + z^2, v = xy + yz + zx, w = x + y + z$, find $J = \frac{\partial(u,v,w)}{\partial(x,y,z)}$.
9. If $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.
10. If $x = r\sin\theta\cos\emptyset, y = r\sin\theta\sin\emptyset, z = r\cos\theta$, show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} = r^2\sin\theta$.
11. Evaluate : $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{1/x}$.
12. Evaluate : $\lim_{x\to a}\left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$.
13. Evaluate : $\lim_{x\to \pi/2}(\sin x)^{\tan(x)}$.
14. Find the extreme values of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$
15. Examine the function $f(x,y) = xy(a - x - y)$ for extreme values.

Module 3

1. Obtain the Reduction formula for $\int \sin^n x\,dx, n > 0$ and hence evaluate $\int_0^{\pi/2} \sin^n x\;dx$.
2. Obtain the Reduction formula for $\int \cos^n x\,dx, n > 0$ and hence evaluate $\int_0^{\pi/2} \cos^n x\;dx$.
3. Obtain the Reduction formula for $\int \sin^m x\cos^n x\;dx$.
4. Evaluate $\int_0^2 x\sqrt{2x - x^2}dx$.
5. Evaluate: $\int_0^{2a} x\sqrt{2ax - x^2}\,dx$.
6. Evaluate : $\int_0^{\pi/6} \sin^2 6x\;\cos^4 6x\,dx$ using Reduction formula.
7. Evaluate : $\int_0^{\pi} \frac{\sin^4\theta}{(1+\cos\theta)^2}\,d\theta$.
8. Evaluate : $\int_0^1 x^{3/2}(1 - x)^{3/2}dx$.
9. Evaluate : $\int_0^{2a} \frac{x^2}{\sqrt{2ax - x^2}}\,dx$
10. Evaluate : $\int_0^{\infty} \frac{x^2}{(1+x^2)^{7/2}}\,dx$
11. Find the orthogonal trajectories of (a) $r^n = a^n\cos n\theta$ (b) $r = a(1 + \sin\theta)$ (c) $r = 2a\;\cos\theta$, *where* a is a parameter.
12. Show that the family of parabolas $y^2 = 4a(x + a)$ is self orthogonal.
13. Find the orthogonal trajectories of the family of ellipses $\frac{x^2}{a^2} + \frac{y^2}{a^2+\lambda} = 1$, where λ is a parameter.
14. Show that the family of curves $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self orthogonal, where λ is a parameter.
15. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes.
16. A body originally at 80°C cools down to 60°C in 20 minutes in the surroundings of temperature 40°C, find the temperature of the body after 40 minutes from the original instant.
17. Water at temperature 10°C takes 5 minutes to warm upto 20°C in a room of temperature 40°C, find the temperature after 20 minutes.

18. A body in air at 25°C, cools from 100°C to 75°C in 1 minute. Find the temperature of the body at the end of 3 minutes.
19. Solve : $(y^3 - 3x^2y)dx - (x^3 - 3xy^2)dy = 0.$
20. Solve : $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$
21. Solve : $xy(1 + xy^2)\frac{dy}{dx} = 1.$
22. Solve : $ye^{xy}\, dx + (xe^{xy} + 2y)dy = 0.$
23. Solve : $\frac{dy}{dx} - \frac{2}{x}y = \frac{y^2}{x^3}.$
24. Solve : $(x^2 + y^2 + x)dx + xydy = 0.$
25. Solve : $(2x\log x - xy)dy + 2ydx = 0.$
26. Solve : $\frac{dy}{dx} = xy^3 - xy.$
27. Solve $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0.$
28. Solve : $x\frac{dy}{dx} + y = x^3y^6.$
29. Solve : $\frac{dy}{dx} + y\tan x = y^3 \sec x.$
30. Solve : $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) dy = 0$

Module 4

1. Find the directional derivative of the function$\emptyset = x^2yz + 4xz^2$ at (1, -2, -1) along 2i-j-2k.
2. Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = grad(x^3 + y^3 + z^3 - 3xyz)$.
3. Find the constants a, b, c such that $\vec{F} = (x + y + az)i + (bx + 2y - z)j + (+cy + 2z)k$ is irrotational. Also find $\emptyset$ such that $\vec{F} = \nabla\emptyset$.
4. Show that $\vec{F} = (y^2 - z^2 + 3yz - 2x)i + (3xz + 2xy)j + (3xy - 2xz + 2z)k$ is both solenoidal and irrotational.
5. Show that $\vec{F} = \frac{xi + yj}{x^2 + y^2}$ is both solenoidal and irrotational.
6. Find the directional derivative of the function$\emptyset = 4xz^3 - 3x^2y^2z$ at (2, -1, 2) along 2i-3j+6k.
7. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point (2, -1, 2).
8. If $\vec{F} = \nabla(xy^3z^2)$ find div $\vec{F}$ and curl $\vec{F}$ at the point (1, -1, 1).
9. If $\vec{F} = (3x^2y - z)i + (xz^3 + y^4)j - (2x^3z^2)k$, find grad(div $\vec{F}$) at (2, -1, 0).
10. If $\vec{F} = (x + y + 1)i + j - (x + y)k, show\ that\ \vec{F}.\left(curl\ \vec{F}\right) = 0.$
11. Show that $\vec{F} = (y + z)i + (z + x)j + (x + y)k$ is irrotational. Also find a scalar function $\emptyset$ such that $\vec{F} = \nabla\emptyset$.
12. Show that $\vec{F} = 2yzi + (x^2z^2 + z\ cos\ yz)j + (2x^2yz + y\cos yz)k$ is a potential field and hence find its scalar potential.
13. Using Green's theorem, find the area enclosed between the parabolas $x^2 = 4ay\ and\ y^2 = 4ax$.
14. Using Green's theorem evaluate : $\int_C (x^2 + xy)dx + (x^2 + y^2)dy,$ where C is the square formed by the lines $x = \pm 1, y = \pm 1$.
15. Using Green's theorem evaluate $\int_C (y - \sin x)dx + \cos x\ dy$, where C is the triangle in the xy – plane formed by the lines y = 0, x = π/2 and y = (2x)/π.

16. Verify Green's theorem for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the boundary of the region enclosed by the lines x = 0, y = 0, x + y = 1.
17. Verify Green's theorem for $\int_C (xy + y^2)dx + x^2dy$, where C is the closed curve made up of the lines y = x and the parabola $y = x^2$.
18. Using Stokes's theorem, evaluate $\int_S (curl\ \vec{f}).\hat{n}\,dS$ for $\vec{f} = (y - z + 2)i + (yz + 4)j - xz)k$ where S is the cubical surface formed by the planes x = 0 , y = 0 , x = 2, y = 0, z = 0.
19. If C is the boundary of the triangle with vertices at P(1, 0, 0), Q(0, 2, 0) and R(0, 0, 3), evaluate $\int_C (x + y)dx + (2x - z)dy + (y + z)dz$ by using Stokes's theorem.
20. Verify Stokes's theorem for $\vec{f} = yi + zj + xk$, for the upper part of the shpere $x^2 + y^2 + z^2 = a^2$.
21. Using the Divergence theorem, evaluate $\int_S \vec{f}.\hat{n}\,dS$, where $\vec{f} = x^3i + y^3j + z^3k$, and S is the surface of the sphere $x^2 + y.^2 + z^2 = a^2$.
22. By using the Divergence theorem, evaluate $\int_S \vec{f}.\hat{n}\,dS$, where $\vec{f} = 4xi \pm 2j + z^2k$, and S is the surface enclosing the region for which $x^2 + y^2 \leq 4\ and\ 0 \leq z \leq 3$.
23. Verify the Divergence theorem for $\vec{f} = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ over the rectangular parallelopiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$.

Module 5

1. Find the rank of the matrix $A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$.
2. Find the rank of the matrix $A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$.
3. Find the rank of the matrix $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$.
4. Find the rank of the matrix $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.
5. Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.
6. Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{bmatrix}$.
7. Test for consistency and solve : $x + y + z = 6; x - y + 2z = 5; 3x + y + z = 8$
8. Test for consistency and solve : $x + 2y + 3z = 14; 4x + 5y + 7z = 35; 3x + 3y + 4z = 21$
9. Test for consistency and solve : $x + 2y + 2z = 1; 2x + y + z = 2; 3x + 2y + 2z = 3; y + z = 0$
10. Find the values of λ for which the system of equations : $x + y + z = 1;\ x + 2y + 4z = \lambda;\ x + 4y + 10z = \lambda^2$ has a solution. Solve it in each case.

11. For what values of λ *and* μ does the system of equations : $x + 2y + 3z = 6; x + 3y + 5z = 9; 2x + 5y + \lambda z = \mu$ has (i) no solution, (ii) unique solution and (iii) infinitely many solutions.
12. Solve by Gauss elimination method : $x + y + z = 4;\ 2x + y - z = 1;\ x - y + 2z = 2.$
13. Solve by Gauss elimination method : $4x + y + z = 4;\ x + 4y - 2z = 4;\ 3x + 2y - 4z = 6.$
14. Solve by Gauss elimination method : $2x - y + 3z = 1;\ -3x + 4y - 5z = 0;\ x + 3y - 6z = 0.$
15. Solve by Gauss elimination method : $x + 2y + z = 3;\ 2x + 3y + 3z = 10;\ 3x - y + 2z = 13.$
16. Solve by Gauss elimination method : $2x + y + 4z = 12;\ 4x + 11y - z = 33;\ 8x - 3y + 2z = 20.$
17. Solve by Gauss-Jordan method : $2x + y + 4z = 12;\ 8x - 3y + 2z = 20; 4x + 11y - z = 33.$
18. Solve by Gauss-Jordan method : $2x - 3y + z = -1;\ x + 4y + 5z = 25; 3x - 4y + z = 2.$
19. Solve by Gauss–Jordan method : $2x + y + z = 10;\ 3x + 2y + 3z = 18;\ x + 4y + 9z = 16.$
20. Solve by Gauss–Jordan method : $x + y + z = 9;\ 2x + y - z = 0;\ 2x + 5y + 7z = 52.$
21. Solve by Gauss -Jordanmethod : $2x + 3y - z = 5;\ 4x + 4y - 3z = 3;\ 2x - 3y + 2z = 2.$
22. Solve by Gauss-Seidel method : $2x + y + 6z = 9; 8x + 3y + 2z = 13; x + 5y + z = 7.$ Carry out 5 iterations to obtain solution correct to 4 decimal places.
23. Solve by Gauss-Seidel method : $20x + y - 2z = 17; 3x + 20y - z = -18; 2x - 3y + 20z = 25.$
24. Solve by Gauss-Seidel method : $10x + y + z = 12; x + 10y + z = 12; x + y + 10z = 12.$
25. Solve by Gauss-Seidel method : $5x + 2y + z = 12; x + 4y + 2z = 15; x + 2y + 5z = 20.$ Take initial approximation to the solution as $[1, 0, 3]^T$.
26. Solve by Gauss-Seidel method : $28x + 4y - z = 32; x + 3y + 10z = 24; 2x + 17y + 4z = 35.$ Carry out 5 iterations to obtain solution correct to 4 decimal places.
27. Using Power method find the largest eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ with $X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Carry out 6 iterations to obtain solution correct to 4 decimal places.
28. Using Power method find the largest eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ with $X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Carry out 6 iterations to obtain solution correct to 4 decimal places.
29. Using Power method find the largest eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ with $X^{(0)} = \begin{bmatrix} 1 \\ 0.8 \\ -0.8 \end{bmatrix}$. Carry out 5 iterations.
30. Using Power method find the largest eigen value and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ with $X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.
31. Diagonalize the matrix $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ and hence find A^4.
32. Diagonalize the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

33. Reduce the matrix $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ to diagonal form. Hence find A^6.

34. Reduce the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ to diagonal form.

35. Reduce the matrix $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$ to diagonal form.