Khaja Bandanawaz University

Faculty of Engineering and Technology

B.E. Second Semester

Question Bank

Subject: Advanced Calculus and Laplace Transforms (19KBMAT21)

MODULE - 1

1. Method of Linear Differential Operator:

(a) Solve : $y'' + 3y' + 2y = 1 + 3x + x^2$.

(b) Solve : $y''' + y'' + 4y' + 4y = x^2 - 4x - 6$

(c) Solve : $(y'' - 4y = \cosh(2x - 1) + 3^x$

(d) Solve : $(D^2 - 2D + 5)y = e^{2x} \sin x$.

(e) Solve : $(D^3 - 1)y = 3 \cos 2x$

(f) Solve : $(D^2 - 4D + 3)y = e^{2x} \cos 3x$.

(g) Solve : $(D^3 - D^2 + 4D - 4)y = \sinh(2x + 3)$.

(h) Solve : $y'' - 3y' + 2y = xe^{3x} + \sin 2x$

(i) Solve : $y''' + 6y'' + 11y' + 6y = e^x + 1$

(j) Solve : $y'' - 6y' + 25y = e^{2x} + \sin x + x$

(k) Solve : $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0$.

(l) Solve : $y''' - 6y'' + 11y' - 6y = 0$

(m) Solve : $(D^3 - 9D^2 + 23D - 15)y = 0$

(n) Solve : $(D^4 - 1)y = 0$

2. Method of Undetermined Coefficients:

(a) Obtain the general solution of : $y'' - 5y' + 6y = e^{3x} + \sin x$.

(b) Solve : $(D^2 + 2D + 4)y = 2x^2 + 3e^{-x}$

(c) Solve : $y'' - 3y' + 2y = x^2 + e^x$.

(d) Solve : $y'' - y' - 2y = x + \sin x$.

(e) Solve : $y'' - 2y' + 3y = x^2 - \cos x$.

(f) Solve : $y'' + 2y' + 4y = 2x^2$.

3. Method of Variation Of Parameters :

(a) Obtain the general solution of $(D^2 + a^2)y = \sec ax$.

(b) Find the general solution of $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$.

(c) Solve : $y'' + 4y = \tan 2x$.

(d) Solve : $(y'' - 2y' + y) = \frac{e^x}{x}$

(e) Solve : $y'' + y = \frac{1}{1+\sin x}$

MODULE – 2

1. Cauchy's Linear differential Equations :

(a) Solve : $x^2y'' + xy' + y = 2cos^2(\log x)$
(b) Solve : $x^2y'' - 3xy' + 4y = (1+x)^2$
(c) Solve : $x^3y''' + 3x^2y'' + xy' + 8y = 65\cos(\log x)$
(d) Solve : $x^2y''3xy' + y = \log x$
(e) Solve : $x^2y'' + xy' + 9y = 3x^2 + \sin(3\log x)$
(f) Solve : $x^2y'' - 4xy' + 6y = \cos(2\log x)$
(g) Solve : $x^3y''' + 3x^2y'' + xy' + y = x + \log x$

2. Legendre's Linear differential equations :

(a) Solve : $(2x-1)^2y'' + (2x-1)y' - 2y = 8x^2 - 2x + 3.$
(b) Solve : $(1+x)^2y'' + (1+x)y' + y = 2\sin(\log(1+x))$
(c) Solve : $(3x+2)^2y'' + 3(3x+2)y' - 36y = 8x^2 + 4x + 1.$
(d) Solve : $(2x+3)^2y'' - (2x+3)y' - 12y = 6x.$
(e) Solve : $(x+2)^2y'' + (x+2)y' + y = \sin(2\log(x+2)).$
(f) Solve : $(2x+1)^2y'' - 6(2x+1)y' + 16y = 8(2x+1)^2.$

3. Non-Linear Differential equations (equations solvable for p,x,y) :

(a) Solve : $y = 2px + p^2y$
(b) Solve : $x^2p^2 + 3xyp + 2y^2 = 0$
(c) Solve : $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}.$
(d) Solve : $y = 2px + y^2p^3$ by solving for x.
(e) Solve : $y(\frac{dy}{dx})^2 + (x-y)\frac{dy}{dx} - x = 0.$
(f) Solve : $y - 2px = \tan^{-1}(xp^2)$
(g) Solve : $xy\left(\frac{dy}{dx}\right)^2 - (x^2+y^2)\frac{dy}{dx} + xy = 0.$
(h) Solve : $x - yp = ap^2$ by solving for x.
(i) Solve : $p^2 + 2py\, cot\, x = y^2$ by solving for p.
(j) Solve : $p(p+y) = x(x+y).$

4. Clairauit's Equation and reducuble to Clairauit's equation, Singular and General solutions:

(a) Solve $(px-y)(py+) = a^2p$ by reducing to Clairauit's form.
(b) Solve $(px-y)(py+) = 2p$ by reducing to Clairauit's form taking the substitution $X=x^2$, $Y=y^2$.
(c) Solve $y^2(y-xp) = x^4p^2$ by reducing to Clairauit's form taking the substitution X=1/x, Y=1/y.
(d) Show that the equation $xp^2 + px - py + 1 - y = 0$ is a Clairauit's equation. Hence obtain the general and singular solutions.
(e) Solve $x^2(y-px) = p^2y$ by reducing to Clairauit's form using the substitution $X=x^2$, $Y=y^2$.
(f) Find the general and singular solution of Clairauit's equation : $y = px + p^2$
(g) Find the general and singular solution of $p = \log(px-y)$
(h) Obtain the general and singular solution of $\sin px \cos y = \cos px \sin y + p.$
(i) Obtain the general and singular solution of $p^2 + 4x^5p - 12x^4y = 0.$
(j) Find the general and singular solution of $y = 2px + p^2y$

MODULE - 3

1. **Formation of partial differential equations** :

(a) $z = y^2 + 2f(\frac{1}{x} + \log y)$.

(b) $f(x + y + z, x^2 + y^2 + z^2) = 0$

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(d) $z = yf(x) + xg(y)$

(e) $f(xy + z^2, x + y + z) = 0$

(f) $f\left(\frac{xy}{z}, z\right) = 0$

(g) $z = f(x + at) + g(x - at)$

(h) $lx + my + nz = \phi(x^2 + y^2 + z^2)$

2. **Solution of Non-Homogeneous PDE by Direct Integration :**

(a) Solve $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, given $\frac{\partial z}{\partial y} = -2 \sin y$, when x = 0, and z = 0 when y is an odd multiple of $\frac{\pi}{2}$.

(b) Solve $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$ subject to the conditions $\frac{\partial z}{\partial x} = \log x$ when y = 1, and z = 0 when x = 1 .

(c) Solve $\frac{\partial^2 z}{\partial x \partial y} = xy$ subject to the conditions $\frac{\partial z}{\partial x} = \log(1 + y)$, when x=1, and z=0 when x=0.

(d) Solve $\frac{\partial^2 u}{\partial x \partial y} = x + y$.

(e) Solve $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y)$.

(f) Solve $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$, u=0 when t=0 and $\frac{\partial u}{\partial t} = 0$ at x=0.

3. **Solution of homogeneous PDE :**

(a) Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$ given that when x = 0, z = $e^y$and $\frac{\partial z}{\partial x} = 1$.

(b) Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} - 4z = 0$, subject to condition z = 1 and $\frac{\partial z}{\partial x} = y$ when x = 0.

(c) Solve $\frac{\partial^2 z}{\partial y^2} = z$ given that when y = 0, , z = e^x and $\frac{\partial z}{\partial y} = e^{-x}$.

(d) Solve $\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$ under the condition z = 0 when x = 0 and, $\frac{\partial z}{\partial x} = a \sin y$ when x = 0.

(e) Solve $\frac{\partial^2 z}{\partial x^2} + 4z = 0$ when x = 0, , z = e^{2y} and $\frac{\partial z}{\partial x} = 2$.

4. **Lagranges Linear PDE :**

(a) Solve : $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$.

(b) Solve : $x(y - z)p + y(z - x)q = z(x - y)$.

(c) Solve :$x^2(y - z)p + y^2(z - x)q = z^2(x - y)$.

(d) Solve: $(x^2 - y^2 - z^2)p + 2xyq = 2xz$.

(e) Solve : $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$.

(f) Solve : $(y + z)p + (z + x)q = x + y$.

(g) Solve : $y^2 zp = x^2(zq + y)$.

(h) Solve : $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

(i) Solve : $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$.

(j) Solve : $x^2p + y^2q = z^2$.

5. One dimensional Heat and wave equations:

(a) Derive one dimensional heat equation.

(b) Derive one dimensional wave equation.

(c) Find the various possible solutions of wave equation by the method of Separation of variables.

(d) Find the various possible solutions of heat equation by the method of Separation of variables.

MODULE - 4

1 mark questions

1. The value of $\int_1^2 \int_1^3 xy^2 dx\, dy$ is ____
2. The value of $\int_0^1 \int_0^2 \int_1^2 x^2yz\, d\, dy\, dz$ is _____
3. $\int_0^\infty e^{-x^2} dx =$______
4. $\Gamma(3.5) =$ ____
5. $\int_0^2 \int_0^x (x+y) dy\, dx =$ ____
6. The integral $2\int_0^\infty e^{-x^2}\, dx$ is ____
7. The value of $\beta(5,3) + \beta(3,5) is$ ____
8. The value of $\Gamma\left(\frac{1}{2}\right) is$ ____
9. The integral $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ after changing the order of integration is _____
10. For a real positive number n, the Gamma function $\Gamma(n) =$ ___
11. The Beta and Gamma functions relation is given by $\beta(m,n) =$ ______
12. In terms of Beta functions $\int_0^{\pi/2} sin^7\theta\sqrt{\cos\theta}\;\, d\theta =$ ____
13. The value of $\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$ is ____
14. Change the order of integration in $\int_0^{4a} \int_{\frac{x^2}{4a}}^{\sqrt{ax}} dy\, dx =$ ___
15. The integral $\iint_R \; f(x,y) dx\, dy$ by changing to polar form becomes ________
16. The integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by changing to polar form becomes ______
17. $\beta\left(3,\frac{1}{2}\right)$ is equal to ___
18. Change the order of integration $\int_0^a \int_0^{2\sqrt{xa}} x^2 dy\, dx, a > 0\; is$ ____

4 marks questions

1. Evaluate $\int_0^{\pi/2} \sqrt{\cos\theta}\, \mathrm{d}\theta$.
2. Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.
3. Express $\int_0^\infty \frac{dx}{1+x^2}$ in terms of Beta function and also evaluate.
4. Evaluate $\int_0^\infty \int_0^x xe^{-x^2/y}\, dy\, dx$.
5. With usual notation show that $\beta(m,n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$
6. Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2} dx\, dy$

7. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$
8. Express $\int_0^1 x^m (1-x^n)^p \; dx$ in terms of Gamma functions.
9. Evaluate $\int_0^1 x^5 (1-x^3)^{10} \; dx$
10. Express the integral $\int_0^1 \frac{dx}{\sqrt{1-x^n}}$ in terms of Gamma function.
11. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^{2/3}}}$.
12. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dx\, dy$.
13. Evaluate $\int_0^2 \int_0^x (x+y)\, dx\, dy$.
14. Evaluate $\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dz\, dy\, dx$
15. Evaluate $\int_0^\infty e^{-x^2} dx$
16. Find $\beta(2,1) + \beta(1,2)$

6 marks questions

1. Evaluate $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)\; dx\, dy\, dz$
2. Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$ and hence evaluate the same.
3. Prove that $\beta\left(m, \frac{1}{2}\right) = 2^{2m-1}\beta(m,m)$
4. Evaluate $\iint xy(x+y) dy\, dx$ taken over the area between $y = x^2 \; and \; y = x$.
5. Show that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$ x $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$
6. Evaluate by changing the order of integration $\int_0^{4a} \int_{\frac{x^2}{4a}}^{\sqrt{ax}} dy\, dx$
7. Evaluate $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2+y^2+z^2) dx\, dy\, dz$
8. Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\, dx$ x $\int_0^1 \frac{1}{\sqrt{1-x^4}} dx = \frac{\pi}{4\sqrt{2}}$
9. By changing the order of integration evaluate $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2) dy\, dx,$ a > 0.
10. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$
11. Express the integral $\int_0^1 \frac{dx}{1-x^n}$ in terms of Gamma function. Hence evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^{2/3}}}$
12. Evaluate $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xy\, dx\, dy$ by changing the order of integration.
13. Show that $\int_{-1}^1 (1+x)^{m-1}(1-x)^{n-1}\, dx = 2^{m+n-1}\, \beta(m,n)$
14. If A is area of rectangular region bounded by the lines x = 0, x = 1 and y = 0, y = 2, evaluate $\int_A (x^2+y^2) dA$
15. Find the volume of the sphere $x^2+y^2+z^2 = a^2$ using triple integration.

MODULE - 5

1. Evaluate the following :
   a) L$\left\{\frac{\cos 2t - \cos 3t}{t}\right\}$
   b) L$\{t^2 e^{-3t} \sin 2t\}$
   c) L$\{e^{-3t}(2\cos 5t - 3\sin 5t\}$
   d) L$\left\{\frac{\cos at - \cos bt}{t}\right\}$

e) $L\left\{2^t + \frac{\cos 2t - \cos 3t}{t} + t\sin t\right\}$

f) $L\{t^3 + 4t^2 - 3t + 5\}$

g) $L\{\cos t \cos 2t \cos 3t\}$

h) $L\{e^{3t}.\sin 5t \sin 3t\}$

i) $L\{te^{-2t} sin^2 t\}$

j) $L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$

2. Laplace Transform of Periodic functions:

a) If $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a - t, & a \le t \le 2a \end{cases}$, $f(t + 2a) = f(t)$, then show that $L\{f(t)\} = \frac{1}{s^2}\tanh\frac{as}{2}$.

b) If a periodic function of period 2a is defined by $f(t) = \begin{cases} E, \ for\ 0 \le t \le a \\ -E, for\ a \le t \le 2a \end{cases}$, where E is a constant, show that $L\{f(t)\} = \frac{E}{s}\tanh as/2$.

c) Find the Laplace transform of full wave rectifier $f(t) = E\ \sin \omega t,\ 0 < t < \pi/\omega$ having period π/ω.

d) Given $f(t) = t^2, 0 < t < 2a$ and $f(t + 2a) = f(t)$, find $L\{f(t)\}$.

e) A periodic function of period $2\pi/\omega$ is defined by $f(t) = \begin{cases} E\ \sin \omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$, where E and ω are positive constants . Show that $L\{f(t)\} = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$.

3. Laplace Transform of Unit Step Functions :

a) Express $f(t) = \begin{cases} 1, 0 < t \le 1 \\ t,\ 1 < t \le 2 \\ t^2,\ t > 2 \end{cases}$ in terms of unit step function and hence find its Laplace transform.

b) Express $f(t) = \begin{cases} \sin t, 0 \le t < \pi \\ \sin 2t,\ \pi \le t < 2\pi \\ \sin 3t,\ t \ge 2\pi \end{cases}$ in terms of unit step function and hence find its Laplace transform.

c) Express $f(t) = \begin{cases} \sin t,\ 0 < t \le \pi/2 \\ \cos t, t > \pi/2 \end{cases}$ in terms of the unit step function and hence find its Laplace transform.

d) Express the function $f(t) = \begin{cases} \pi - t, 0 < t \le \pi \\ \sin t,\ t > \pi \end{cases}$ in terms of the unit step function and find its Laplace transform.

e) Express the function $f(t) = \begin{cases} t^2, 0 < t \le 2 \\ 4t,\ t > 2 \end{cases}$ in terms of the Heaviside unit step function and find its Laplace transform.

4. Solve by using Laplace transform method :

a) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t},\ y(0) = 0, y'(0) = 0.$

b) $y''' + 2y'' - y' - 2y = 0,\ y(0) = y'(0) = 0, y''(0) = 6.$

c) $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^{2t},\ x(0) = 0, x'(0) = -1.$

d) $y'' + 6y' + 9y = 12t^2e^{-3t},\ y(0) = 0 = y'(0).$

e) $y'' + 3y' + 2y = 0,\ y(0) = 1, y'(0) = 0.$

5. Find the Inverse Laplace transform of

a) $\frac{4s+5}{(s+1)^2(s+2)}$

b) $\frac{s+3}{s^2-4s+13}$

c) $\log\frac{s^2+1}{s(s+1)}$

d) $\frac{7s+4}{4s^2+4s+9}$

e) $\frac{2s-1}{s^2+2s+17}$

f) $\frac{s+5}{s^2-6s+13}$

g) $\frac{7s}{4s^2+4s+9}$

6. Inverse Laplace Transform using Convolution Theorem :

   a) $\frac{1}{s(s^2+a^2)}$

   b) $\frac{s}{(s^2+a^2)^2}$

   c) $\frac{1}{(s^2+a^2)^2}$

   d) $\frac{1}{(s-1)(s^2+1)}$

   e) $\frac{s}{(s-1)(s^2+4)}$